

On a diophantine inequality with prime numbers of a special type

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Abstract

We consider the Diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < (\log N)^{-E},$$

where $1 < c < \frac{15}{14}$, N is a sufficiently large real number and $E > 0$ is an arbitrarily large constant. We prove that the above inequality has a solution in primes p_1, p_2, p_3 such that each of the numbers $p_1 + 2, p_2 + 2, p_3 + 2$ has at most $\left\lceil \frac{369}{180-168c} \right\rceil$ prime factors, counted with the multiplicity.

1 Introduction and statement of the result

We consider the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < \Delta, \tag{1}$$

where $c > 1$ is a constant, N is a sufficiently large real number and $\Delta = \Delta(N)$ is a function such that $\Delta(N) \rightarrow 0$ as $N \rightarrow \infty$. Having in mind I. M. Vinogradov's famous theorem about Goldbach's ternary problem (see [21]), one may expect that if c is not much greater than 1 and $\Delta(N)$ is a suitable function, then inequality (1) has a solution in prime numbers p_1, p_2, p_3 . A result of this type with $1 < c < \frac{15}{14}$ and with $\Delta = N^{-\kappa}$ for certain $\kappa = \kappa(c) > 0$ was established in 1992 by the author [16]. Several improvements were made since then and the strongest of them is due to Baker and Weingartner [1]. In 2014 they established that (1) is solvable in primes, provided that N is large enough, $1 < c < \frac{10}{9}$ and $\Delta = N^{-\kappa}$ for certain $\kappa = \kappa(c) > 0$.

Suppose that r is a natural number and let \mathcal{P}_r be the set of positive integers having at most r prime factors, counted with the multiplicity. (We say that the numbers from \mathcal{P}_r are almost primes of order r .) In 1973 Chen [3], improving results of other mathematicians, established that there exist infinitely many primes p such that $p + 2 \in \mathcal{P}_2$. Bearing in mind Chen's result, one may try to study the arithmetical properties of the

set of primes p such that $p + 2 \in \mathcal{P}_r$ for a fixed $r \geq 2$ and, in particular, to establish the solvability of diophantine equations or inequalities in such primes. For example, Matomäki and Shao [11], improving author's results from [17] and [18] as well as a result of Matomäki [10], proved that every sufficiently large odd integer N can be represented as a sum of three primes p_1, p_2, p_3 such that $p_i + 2 \in \mathcal{P}_2$, $i = 1, 2, 3$. Other results of this type were found by the author [19], and by Dimitrov and Todorova [6].

One may expect that if the constant $c > 1$ is close to one then inequality (1), with a suitable Δ satisfying $\Delta \rightarrow 0$ as $N \rightarrow \infty$, is solvable in primes p_i such that $p_i + 2$ are almost primes of certain fixed order. An attempt to establish a result of this type was made by Dimitrov [5], but he considers the inequality (1) only when $c < \frac{4}{21}$, whilst the case $c > 1$ is more interesting. Dimitrov recently announced that he is able to prove the solvability of (1) in the case $1 < c < \frac{121}{120}$ but such a result has not been published.

In the present paper we assume that c is a constant such that

$$1 < c < \frac{15}{14}. \quad (2)$$

We consider the inequality (1) with

$$\Delta = (\log N)^{-E}, \quad (3)$$

where $E > 0$ is an arbitrarily large constant and we prove the following

Theorem 1. *Let c be a constant satisfying (2) and let N be a sufficiently large real number. Then inequality (1), with Δ specified by (3), has a solution in primes p_1, p_2, p_3 such that each of the numbers $p_i + 2$, $i = 1, 2, 3$ has at most $\left[\frac{369}{180-168c}\right]$ prime factors, counted with the multiplicity.*

It follows from Theorem 1 that if $c > 1$ is close to 1, then inequality (1), with Δ given by (3), has a solution in primes p_i such that $p_i + 2 \in \mathcal{P}_{30}$.

We can establish a similar result for the inequality (1) with $\Delta = N^{-\kappa}$ for certain $\kappa > 0$, but then we would have $p_i + 2 \in \mathcal{P}_m$, $i = 1, 2, 3$, where m depends on c and κ .

Notations in the paper shall be as follows. By ε and A we denote an arbitrarily small positive number and respectively, an arbitrarily large constant which may not be the same in different formulae. The letter p always denotes a prime number. By $\tau(n)$, $\mu(n)$, $\varphi(n)$ and $\Lambda(n)$ we denote the number of divisors of n , Möbius' function, Euler's function and Von Mangoldt's function respectively. We shall use (m, n) and $[m, n]$ for the greatest common divisor and the least common multiple of the integers m, n . (We denote in this way also open and closed intervals from the real line, but the meaning will be clear from the context). Let $[t]$ be the integer part of the real number t and $e(t) = e^{2\pi it}$. With χ we denote a Dirichlet's character. As usual, $\sum_{\chi \pmod{q}}$ means that the summation is taken over all Dirichlet's characters modulo q . Respectively, $\sum_{\chi \pmod{q}^*}$ means that the summation is taken over the primitive Dirichlet's characters modulo q .

Suppose that χ is a Dirichlet's character and $L(s, \chi)$ is the corresponding L -function. If $T \geq 2$ and $0 \leq \sigma \leq 1$ we denote by $N(T, \sigma, \chi)$ the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ such that $|\gamma| \leq T$ and $\sigma \leq \beta \leq 1$. We also write $N(T, \chi) = N(T, 0, \chi)$.

By

$$\psi(y) = \sum_{n \leq y} \Lambda(n), \quad \psi(y, k, l) = \sum_{\substack{n \leq y \\ n \equiv l \pmod{k}}} \Lambda(n) \quad (4)$$

we denote Chebyshev's functions and we define $\Delta(y, k, l)$ by

$$\Delta(y, k, l) = \psi(y, k, l) - \frac{y}{\varphi(k)}. \quad (5)$$

For a given Dirichlet's character χ we write

$$\psi(y, \chi) = \sum_{n \leq y} \Lambda(n) \chi(n). \quad (6)$$

Finally, by \square we mark an end of a proof or its absence.

2 Beginning of the proof

Let η, δ, ξ, μ be positive real numbers depending on c . We shall specify them later but for now only assume that they satisfy the conditions

$$\xi + 3\delta < \frac{12}{25}, \quad 2 < \frac{\delta}{\eta} < 3, \quad \mu < 1. \quad (7)$$

We define

$$X = N^{\frac{1}{c}}, \quad z = X^\eta, \quad D = X^\delta, \quad \tau = X^{\xi-c}, \quad (8)$$

$$r = [(\log X)^2], \quad \Xi = (\log X)^{E+3} \quad (9)$$

and

$$P(z) = \prod_{2 < p < z} p, \quad (10)$$

where the product is taken over prime numbers.

Consider the sum

$$\Gamma = \sum_{\substack{\mu X < p_1, p_2, p_3 \leq X \\ (1), (12)}} (\log p_1)(\log p_2)(\log p_3), \quad (11)$$

where the summation is taken over the primes p_1, p_2, p_3 from the interval $(\mu X, X]$ which satisfy (1) (with Δ given by (3)), as well as the conditions

$$(p_1 + 2, P(z)) = (p_2 + 2, P(z)) = (p_3 + 2, P(z)) = 1. \quad (12)$$

If we prove the inequality

$$\Gamma > 0, \quad (13)$$

then the equation (1) would have a solution in primes p_1, p_2, p_3 satisfying (12). If the number $p_i + 2$ has l prime factors counted with multiplicity, then from (8), (10), (12) and from the condition $\mu X < p_i \leq X$ we easily find that

$$l \leq \frac{1}{\eta}. \quad (14)$$

This means that $p_i + 2$ would be an almost-prime of order $[\eta^{-1}]$. Therefore, to prove the theorem we have to establish (13) for a suitable choice of η .

Firstly, we use the following

Lemma 2. *Let a, δ be real numbers, $0 < \delta < \frac{a}{4}$, and let r be a positive integer. There exists a function $\theta(y)$ which is r times continuously differentiable and such that*

$$\begin{aligned} \theta(y) &= 1 & \text{for } |y| \leq a - \delta, \\ 0 < \theta(y) &< 1 & \text{for } a - \delta < |y| \leq a + \delta, \\ \theta(y) &= 0 & \text{for } |y| \geq a + \delta, \end{aligned}$$

and its Fourier transform

$$\Theta(x) = \int_{-\infty}^{\infty} \theta(y) e(-xy) dy \quad (15)$$

satisfies the inequality

$$|\Theta(x)| \leq \min \left(2a, \frac{1}{|x|} \left(\frac{r}{|x|\delta} \right)^r \right). \quad (16)$$

Proof. This is an old result which goes back to Segal [13]. □

We apply this lemma with r given by (9) and with $a = \frac{7\Delta}{8}$, $\delta = \frac{\Delta}{8}$, where Δ is specified by (3). Hence we have

$$\theta(y) = 0 \quad \text{for } |y| \geq \Delta, \quad 0 < \theta(y) \leq 1 \quad \text{for } |y| < \Delta \quad (17)$$

and from (11) and (17) we find that

$$\Gamma \geq \Gamma' := \sum_{\substack{\mu X < p_1, p_2, p_3 \leq X \\ (12)}} (\log p_1)(\log p_2)(\log p_3) \theta(p_1^c + p_2^c + p_3^c - N) \quad (18)$$

For $i = 1, 2, 3$ we consider the quantities

$$\Lambda_i = \sum_{d|(p_i+2, P(z))} \mu(d) = \begin{cases} 1 & \text{if } (p_i + 2, P(z)) = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (19)$$

Bearing in mind (12) and (18) we find that

$$\Gamma' = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1 \Lambda_2 \Lambda_3 \theta(p_1^c + p_2^c + p_3^c - N). \quad (20)$$

Now we apply the following fundamental result from sieve theory

Lemma 3. Suppose that $D > 4$ is a real number. There exist arithmetical functions $\lambda^\pm(d)$ (called Rosser's functions of level D) with the following properties.

1) For any positive integer d we have

$$|\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \quad \text{if } d > D \quad \text{or} \quad \mu(d) = 0. \quad (21)$$

2) If n is a positive integer then

$$\sum_{d|n} \lambda^-(d) \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda^+(d).$$

3) If z is a real number such that $z^2 \leq D \leq z^3$ and if

$$P(z) = \prod_{2 < p < z} p, \quad \mathfrak{P} = \prod_{2 < p < z} \left(1 - \frac{1}{p-1}\right), \quad \mathfrak{N}^\pm = \sum_{d|P(z)} \frac{\lambda^\pm(d)}{\varphi(d)}, \quad s_0 = \frac{\log D}{\log z}, \quad (22)$$

then we have

$$\mathfrak{P} \leq \mathfrak{N}^+ \leq \mathfrak{P} \left(F(s_0) + O\left((\log D)^{-\frac{1}{3}}\right) \right), \quad (23)$$

$$\mathfrak{P} \geq \mathfrak{N}^- \geq \mathfrak{P} \left(f(s_0) + O\left((\log D)^{-\frac{1}{3}}\right) \right), \quad (24)$$

where $F(s)$ and $f(s)$ (the functions of the linear sieve) satisfy

$$f(s) = 2e^\gamma s^{-1} \log(s-1), \quad F(s) = 2e^\gamma s^{-1} \quad \text{for } 2 \leq s \leq 3. \quad (25)$$

Here γ stands for the Euler constant.

Proof. This is a special case of a more general result — see Greaves [7, Ch. 4]. □

Suppose that $\lambda^\pm(d)$ are the Rosser functions of level D , where D is defined by (8). According to Lemma 3, if we denote

$$\Lambda_i^\pm = \sum_{d|(p_i+2, P(z))} \lambda^\pm(d), \quad i = 1, 2, 3 \quad (26)$$

and if Λ_i is given by (19), then we have

$$\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+, \quad \Lambda_i \in \{0, 1\}. \quad (27)$$

Next we use the following

Lemma 4. Suppose that Λ_i, Λ_i^\pm , $i = 1, 2, 3$ are real numbers satisfying (27). Then we have

$$\Lambda_1 \Lambda_2 \Lambda_3 \geq \Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+. \quad (28)$$

Proof. The proof is elementary and similar to the proof of [2, Lemma 13]). \square

We apply (20), (27) and (28) and then we substitute the quantity from the right side of (28) for $\Lambda_1 \Lambda_2 \Lambda_3$ in (20). We find that

$$\Gamma' \geq \Gamma_1 + \Gamma_2 + \Gamma_3 - 2\Gamma_4,$$

where $\Gamma_1, \dots, \Gamma_4$ are the contributions coming from the consecutive terms of the right side of (28). It is clear that

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \theta(p_1^c + p_2^c + p_3^c - N), \quad (29)$$

$$\Gamma_4 = \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \theta(p_1^c + p_2^c + p_3^c - N). \quad (30)$$

Hence, we get

$$\Gamma' \geq 3\Gamma_1 - 2\Gamma_4. \quad (31)$$

Consider Γ_1 . (The study of Γ_4 is likewise). We apply Fourier's inversion formula

$$\theta(t) = \int_{-\infty}^{\infty} \Theta(x) e(xt) dx$$

as well as (26) to find that

$$\begin{aligned} \Gamma_1 &= \sum_{\mu X < p_1, p_2, p_3 \leq X} (\log p_1)(\log p_2)(\log p_3) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \int_{-\infty}^{\infty} \Theta(x) e(x(p_1^c + p_2^c + p_3^c - N)) dx \\ &= \int_{-\infty}^{\infty} \Theta(x) L^-(x) L^+(x)^2 e(-Nx) dx, \end{aligned} \quad (32)$$

where

$$L^\pm(x) = \sum_{\mu X < p \leq X} (\log p) e(xp^c) \sum_{d|(p+2, P(z))} \lambda^\pm(d).$$

Changing the order of summation, we get

$$L^\pm(x) = \sum_{d|P(z)} \lambda^\pm(d) \sum_{\substack{\mu X < p \leq X \\ d|p+2}} (\log p) e(xp^c). \quad (33)$$

We divide the integral from (32) into three parts as follows:

$$\Gamma_1 = \Gamma_1^{(1)} + \Gamma_1^{(2)} + \Gamma_1^{(3)}, \quad (34)$$

where

$$\Gamma_1^{(1)} = \int_{|x| < \tau} \Theta(x) L^-(x) L^+(x)^2 e(-Nx) dx, \quad (35)$$

$$\Gamma_1^{(2)} = \int_{\tau < |x| < \Xi} \Theta(x) L^-(x) L^+(x)^2 e(-Nx) dx, \quad (36)$$

$$\Gamma_1^{(3)} = \int_{|x| > \Xi} \Theta(x) L^-(x) L^+(x)^2 e(-Nx) dx. \quad (37)$$

Similarly, for the quantity Γ_4 defined by (30) we find

$$\Gamma_4 = \Gamma_4^{(1)} + \Gamma_4^{(2)} + \Gamma_4^{(3)}, \quad (38)$$

where

$$\Gamma_4^{(1)} = \int_{|x| < \tau} \Theta(x) L^+(x)^3 e(-Nx) dx, \quad (39)$$

$$\Gamma_4^{(2)} = \int_{\tau < |x| < \Xi} \Theta(x) L^+(x)^3 e(-Nx) dx, \quad (40)$$

$$\Gamma_4^{(3)} = \int_{|x| > \Xi} \Theta(x) L^+(x)^3 e(-Nx) dx. \quad (41)$$

It is easy to estimate $\Gamma_1^{(3)}$ and $\Gamma_4^{(3)}$. It is clear from (33) that $L^\pm(x) \ll X^{1+\varepsilon}$. We also use (3), (9) and (16) to find that

$$\Gamma_1^{(3)}, \Gamma_4^{(3)} \ll X^{3+\varepsilon} \left(\frac{r}{\Delta}\right)^r \int_{\Xi}^{\infty} \frac{dx}{x^{r+1}} \ll X^{3+\varepsilon} \left(\frac{r}{\Delta \Xi}\right)^r \ll 1. \quad (42)$$

From (18), (31), (34) and (38) it follows that

$$\Gamma \geq \left| 3\Gamma_1^{(1)} - 2\Gamma_4^{(1)} \right| - c_0 \left(|\Gamma_1^{(2)}| + |\Gamma_4^{(2)}| + 1 \right), \quad (43)$$

where $c_0 > 0$ is an absolute constant.

3 The integrals $\Gamma_1^{(1)}$ and $\Gamma_4^{(1)}$

In this section we find asymptotic formulae for $L^\pm(x)$, provided that $|x| < \tau$. The arithmetic structure of the Rosser weights $\lambda^\pm(d)$ is not important here, so we consider a sum of the form

$$L(x) = \sum_{d \leq D} \lambda(d) \sum_{\substack{\mu X < p \leq X \\ d|p+2}} (\log p) e(xp^c), \quad (44)$$

and we assume that $\lambda(d)$ are real numbers satisfying

$$|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \quad \text{if} \quad 2 \mid d \quad \text{or} \quad \mu(d) = 0. \quad (45)$$

We also define

$$I(x) = \int_{\mu X}^x e(xt^c) dt, \quad (46)$$

To study $I(x)$, we need the following

Lemma 5. *Consider the integral*

$$I = \int_a^b G(x) e(F(x)) dx$$

where $G(x), F(x)$ are real functions with continuous second derivatives. Assume that the function $G(x)/F'(x)$ is monotonous and suppose that $|G(x)| \leq H$ for all $x \in [a, b]$.

If $|F'(x)| \geq h > 0$ for all $x \in [a, b]$ then $I \ll Hh^{-1}$.

If $|F''(x)| \geq h > 0$ for all $x \in [a, b]$ then $I \ll Hh^{-\frac{1}{2}}$.

Proof. See [14, p. 71]. □

We also need Bombieri–Vinogradov’s theorem:

Lemma 6. *Suppose that $x > 2$, $Q > 2$ and consider the sum*

$$\Sigma = \sum_{d \leq Q} \max_{y \leq x} \max_{(l, d)=1} |\Delta(y, d, l)|,$$

where $\Delta(y, d, l)$ is defined by (5). For any constant $A > 0$ there exists $B = B(A) > 0$ such that if $Q \leq \sqrt{x} (\log x)^{-B}$, then $\Sigma \ll x(\log x)^{-A}$.

Proof. See [4, Ch. 28]. □

In the next lemma we give explicit formulae for Chebyshev’s function $\psi(y)$ and for the function $\psi(y, \chi)$ defined by (6).

Lemma 7. Suppose that $2 \leq T \leq y$.

We have

$$\psi(y) = y - \sum_{|\gamma| < T} \frac{y^\rho}{\rho} + O\left(\frac{y \log y}{T}\right), \quad (47)$$

where the summation runs over the non-trivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta function such that $|\gamma| < T$.

If $r > 1$ and χ is a primitive character (mod r), then we have

$$\psi(y, \chi) = - \sum_{|\gamma| < T} \frac{y^\rho}{\rho} + \sum_{|\gamma| < 1} \frac{1}{\rho} + O\left(\frac{y \log(ry)}{T}\right), \quad (48)$$

where the summation is taken over the non-trivial zeros $\rho = \beta + i\gamma$ of Dirichlet's L -function $L(s, \chi)$ such that $|\gamma| < T$ and $|\gamma| < 1$, respectively.

Proof. See [4, Ch. 17] and [4, Ch. 19] □

The next lemma provides an information about the density of the zeroes of Dirichlet's L -functions.

Lemma 8. If χ is a primitive Dirichlet's character modulo d and if $T \geq 2$, then we have

$$N(T, \chi) \ll T \log(dT). \quad (49)$$

If $Q \geq 1$ and $T \geq 2$, then for the sum

$$\Sigma(T, \sigma, Q) = \sum_{d \leq Q} \sum_{\chi \pmod{d}^*} N(T, \sigma, \chi) \quad (50)$$

we have

$$\Sigma(T, \sigma, Q) \ll \begin{cases} (Q^2 T)^{\frac{3(1-\sigma)}{2-\sigma}} (\log(QT))^9 & \text{if } \frac{1}{2} \leq \sigma \leq \frac{4}{5}, \\ (Q^2 T)^{\frac{2(1-\sigma)}{\sigma}} (\log(QT))^{14} & \text{if } \frac{4}{5} \leq \sigma \leq 1. \end{cases} \quad (51)$$

Proof. The proof of (49) can be found in [4, Ch. 16] and for the proof of (51) see [12, Theorem 12.2]. □

In the next lemma we present an analog of the estimate from [16, Lemma 5].

Lemma 9. Consider the sum

$$\mathfrak{L}(T, Q, X) = \sum_{d \leq Q} \sum_{\chi \pmod{d}^*} \sum_{|\gamma| \leq T} X^\beta, \quad (52)$$

where $X \geq 2$, $Q \geq 1$, $T \geq 2$ and where the summation in the inner sum is taken over the non-trivial zeros $\rho = \beta + i\gamma$ of Dirichlet's L -function $L(s, \chi)$ such that $|\gamma| \leq T$.

Suppose that

$$Q^2 T \leq X^{\frac{12}{25}}. \quad (53)$$

Then we have

$$\mathfrak{L}(T, Q, X) \ll (\log X)^{20} \left(X + X^{\frac{4}{5}} Q T^{\frac{1}{2}} \right). \quad (54)$$

Proof. The proof of (54) is standard but for reader's convenience we present the arguments. Suppose that χ is a primitive character modulo d and $\rho = \beta + i\gamma$ is a non-trivial zero of $L(s, \chi)$. We start with the identity

$$X^\beta = 1 + (\log X) \int_0^1 X^\sigma \Phi(\sigma, \beta) d\sigma, \quad (55)$$

where $\Phi(\sigma, \beta) = 0$ for $\sigma > \beta$ and $\Phi(\sigma, \beta) = 1$ for $\sigma \leq \beta$. It is clear that

$$\sum_{|\gamma| \leq T} \Phi(\sigma, \beta) = N(T, \sigma, \chi),$$

hence applying (55) and the estimate (49) from Lemma 8 we find that

$$\begin{aligned} \sum_{|\gamma| \leq T} X^\beta &= N(T, \chi) + (\log X) \int_0^1 X^\sigma N(T, \sigma, \chi) d\sigma \\ &\ll X^{\frac{1}{2}} T (\log X)^2 + (\log X) \int_{\frac{1}{2}}^1 X^\sigma N(T, \sigma, \chi) d\sigma. \end{aligned}$$

From the above formula, (50) and (52) we find that

$$\mathfrak{L}(T, Q, X) \ll X^{\frac{1}{2}} Q^2 T (\log X)^2 + (\log X) \int_{\frac{1}{2}}^1 X^\sigma \Sigma(T, \sigma, Q) d\sigma.$$

We apply the estimate (51) from Lemma 8 and find that

$$\mathfrak{L}(T, Q, X) \ll (\log X)^{15} \left(X^{\frac{1}{2}} Q^2 T + \mathcal{I}_1 + \mathcal{I}_2 \right), \quad (56)$$

where

$$\mathcal{I}_1 = \int_{\frac{1}{2}}^{\frac{4}{5}} X^\sigma (Q^2 T)^{\frac{3(1-\sigma)}{2-\sigma}} d\sigma, \quad \mathcal{I}_2 = \int_{\frac{4}{5}}^1 X^\sigma (Q^2 T)^{\frac{2(1-\sigma)}{\sigma}} d\sigma.$$

To estimate the integral \mathcal{I}_1 , we write it in the form

$$\mathcal{I}_1 = \int_{\frac{1}{2}}^{\frac{4}{5}} e^{h_1(\sigma)} d\sigma, \quad h_1(\sigma) = \sigma(\log X) + \frac{3(1-\sigma)}{2-\sigma} \log(Q^2 T).$$

Using the condition (53), it is easy to verify that $h'_1(\sigma) \geq 0$ for $\frac{1}{2} \leq \sigma \leq \frac{4}{5}$. This means that $\max_{\frac{1}{2} \leq \sigma \leq \frac{4}{5}} h_1(\sigma) = h_1\left(\frac{4}{5}\right)$ and therefore

$$\mathcal{I}_1 \ll e^{h_1\left(\frac{4}{5}\right)} = X^{\frac{4}{5}} Q T^{\frac{1}{2}}. \quad (57)$$

Consider \mathcal{I}_2 . We have

$$\mathcal{I}_2 = \int_{\frac{4}{5}}^1 e^{h_2(\sigma)} d\sigma, \quad h_2(\sigma) = \sigma(\log X) + \frac{2(1-\sigma)}{\sigma} \log(Q^2 T).$$

We have $h_2''(\sigma) > 0$ for $\frac{4}{5} \leq \sigma \leq 1$, hence $\max_{\frac{4}{5} \leq \sigma \leq 1} h_2(\sigma) = \max\left(h_2\left(\frac{4}{5}\right), h_2(1)\right)$ and therefore

$$\mathcal{I}_2 \ll e^{h_2\left(\frac{4}{5}\right)} + e^{h_2(1)} = X^{\frac{4}{5}} Q T^{\frac{1}{2}} + X. \quad (58)$$

The estimate (54) is a consequence of (53) and (56) – (58). □

We shall prove the following

Lemma 10. *Suppose that D and τ are defined by (8) and that ξ and δ satisfy (7). If $L(x)$ and $I(x)$ are defined by (44) and (46) and if $|x| < \tau$ then we have*

$$L(x) = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(x) + O\left(X(\log X)^{-A}\right), \quad (59)$$

where $A > 0$ is an arbitrarily large constant.

Proof. One may easily see that

$$L(x) = L_1(x) + O\left(X^{\frac{1}{2}+\varepsilon}\right), \quad (60)$$

where

$$L_1(x) = \sum_{d \leq D} \lambda(d) S(x, d), \quad S(x, d) = \sum_{\substack{\mu X < n \leq X \\ d|n+2}} \Lambda(n) e(xn^c). \quad (61)$$

To find an asymptotic formula for $L_1(x)$, we shall proceed in different ways according to the size of $|x|$.

Firstly, consider the case

$$|x| \leq X^{-c}(\log X)^B, \quad (62)$$

where $B > 0$ is a constant which we shall specify later. For the sum $S(x, d)$, defined by (61), we apply Abel's formula and (4) to find that

$$\begin{aligned}
S(x, d) &= e(xX^c) \sum_{\substack{\mu X < n \leq X \\ d|n+2}} \Lambda(n) - \int_{\mu X}^X \sum_{\substack{\mu X < n \leq t \\ d|n+2}} \Lambda(n) \frac{d}{dt} e(xt^c) dt \\
&= e(xX^c) (\psi(X, d, -2) - \psi(\mu X, d, -2)) \\
&\quad - \int_{\mu X}^X (\psi(t, d, -2) - \psi(\mu X, d, -2)) \frac{d}{dt} e(xt^c) dt.
\end{aligned}$$

Now we use (5) to get

$$\begin{aligned}
S(x, d) &= e(xX^c) \left(\frac{X - \mu X}{\varphi(d)} + \Delta(X, d, -2) - \Delta(\mu X, d, -2) \right) \\
&\quad - \int_{\mu X}^X \left(\frac{t - \mu X}{\varphi(d)} + \Delta(t, d, -2) - \Delta(\mu X, d, -2) \right) \frac{d}{dt} e(xt^c) dt.
\end{aligned}$$

Therefore, using (46) and the assumption (62) we find that

$$\begin{aligned}
S(x, d) &= \frac{1}{\varphi(d)} \left(e(xX^c) (X - \mu X) - \int_{\mu X}^X (t - \mu X) \frac{d}{dt} e(xt^c) dt \right) \\
&\quad + O \left((1 + |x|X^c) \max_{y \leq X} \max_{(l, d)=1} |\Delta(y, d, l)| \right) \\
&= \frac{I(x)}{\varphi(d)} + O \left((\log X)^B \max_{y \leq X} \max_{(l, d)=1} |\Delta(y, d, l)| \right). \tag{63}
\end{aligned}$$

It remains to substitute the above expression for $S(x, d)$ in the formula for $L_1(x)$ in (61). Since $\delta < \frac{1}{2}$, we are in position to apply Lemma 6 and find that the contribution from the remainder term in (63) is $\ll X(\log X)^{-A}$. We also take into account (60) to conclude that in the case (62) the asymptotic formula (59) is true.

Consider now the case

$$X^{-c}(\log X)^B < |x| \leq \tau. \tag{64}$$

We proceed with the sum $S(x, d)$, specified by (61), in a different way. Using the prop-

erties of Dirichlet's characters we get

$$\begin{aligned} S(x, d) &= \sum_{\mu X < n \leq X} \Lambda(n) e(xn^c) \frac{1}{\varphi(d)} \sum_{\chi \pmod{d}} \chi(n) \bar{\chi}(-2) \\ &= \frac{1}{\varphi(d)} \sum_{\chi \pmod{d}} \bar{\chi}(-2) \sum_{\mu X < n \leq X} \Lambda(n) \chi(n) e(xn^c). \end{aligned}$$

We separate the contribution from the principal character to get

$$S(x, d) = \frac{1}{\varphi(d)} Y(X) + O\left(\frac{(\log X)^2}{\varphi(d)}\right) + \mathfrak{N}, \quad (65)$$

where

$$Y(X) = \sum_{\mu X < n \leq X} \Lambda(n) e(xn^c) \quad (66)$$

and where \mathfrak{N} comes from the non-principal characters. We express \mathfrak{N} as a sum over primitive characters and find that

$$\mathfrak{N} = \frac{1}{\varphi(d)} \sum_{\substack{r|d \\ r>1}} \sum_{\chi \pmod{r}^*} \bar{\chi}(-2) \sum_{\substack{\mu X < n \leq X \\ (n,d)=1}} \Lambda(n) \chi(n) e(xn^c).$$

If we omit the condition $(n, d) = 1$ imposed in the sum over n , then the resulting error will be $O((\log X)^2)$. (We leave the easy verification to the reader). We use (64) and substitute the expression for $S(x, d)$ in the sum $L_1(x)$ from (61). Taking into account (60) we find that

$$L(x) = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} Y(X) + O\left(X^{\frac{1}{2}+\varepsilon}\right) + \mathfrak{M}, \quad (67)$$

where

$$\mathfrak{M} = \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} \sum_{\substack{r|d \\ r>1}} \sum_{\chi \pmod{r}^*} \bar{\chi}(-2) Y(X, \chi),$$

and

$$Y(X, \chi) = \sum_{\mu X < n \leq X} \Lambda(n) \chi(n) e(xn^c). \quad (68)$$

We change the order of summation to get

$$\mathfrak{M} = \sum_{1 < r \leq D} \left(\sum_{\substack{d \leq D \\ r|d}} \frac{\lambda(d)}{\varphi(d)} \right) \sum_{\chi \pmod{r}^*} \bar{\chi}(-2) Y(X, \chi).$$

Using (45), we easily find that

$$\sum_{\substack{d \leq D \\ r|d}} \frac{\lambda(d)}{\varphi(d)} \ll \frac{\log X}{\varphi(r)},$$

hence

$$\mathfrak{M} \ll (\log X) \sum_{1 < r \leq D} \frac{1}{\varphi(r)} \sum_{\chi \pmod{r}^*} |Y(X, \chi)|. \quad (69)$$

Consider the sum $Y(X, \chi)$. We apply Abel's formula and use (6) to find that

$$\begin{aligned} Y(X, \chi) &= e(xX^c) \sum_{\mu X < n \leq X} \Lambda(n) \chi(n) - \int_{\mu X}^X \left(\sum_{\mu X < n \leq t} \Lambda(n) \chi(n) \right) \frac{d}{dt} e(xt^c) dt \\ &= e(xX^c) (\psi(X, \chi) - \psi(\mu X, \chi)) - \int_{\mu X}^X (\psi(t, \chi) - \psi(\mu X, \chi)) \frac{d}{dt} e(xt^c) dt. \end{aligned}$$

We choose

$$T = |x|X^c D (\log X)^B. \quad (70)$$

From (7), (8) and (64) we easily see that $2 \leq T \leq \mu X$. Now we apply formula (48) from Lemma 7 and find that

$$\begin{aligned} Y(X, \chi) &= e(xX^c) \left(- \sum_{|\gamma| < T} \frac{X^\rho - (\mu X)^\rho}{\rho} + O\left(\frac{X(\log X)^2}{T}\right) \right) \\ &\quad + \int_{\mu X}^X \left(\sum_{|\gamma| < T} \frac{t^\rho - (\mu X)^\rho}{\rho} + O\left(\frac{X(\log X)^2}{T}\right) \right) \frac{d}{dt} e(xt^c) dt. \end{aligned}$$

We estimate the contributions from the error terms, then we change the order of summation and integration and finally we integrate by parts to get

$$Y(X, \chi) = - \sum_{|\gamma| < T} I_\rho(x) + O\left(\frac{X}{T} (\log X)^2 (1 + |x|X^c)\right),$$

where

$$I_\rho(x) = \int_{\mu X}^X t^{\rho-1} e(xt^c) dt. \quad (71)$$

Now, we substitute the last expression for $Y(X, \chi)$ in (69) and use (64) to obtain

$$\mathfrak{M} \ll (\log X) \sum_{1 < r \leq D} \frac{1}{\varphi(r)} \sum_{\chi \pmod{r}^*} \sum_{|\gamma| \leq T} |I_\rho(x)| + T^{-1} |x| X^{1+c} D (\log X)^3. \quad (72)$$

We study the sum $Y(X)$, defined by (66), in the same manner but now we apply formula (47) from Lemma 7. We take the parameter T , defined by (70), and after some calculations we find that

$$Y(X) = I(x) - \sum_{|\gamma| < T} I_\rho(x) + O\left(T^{-1} |x| X^{1+c} (\log X)^3\right),$$

where $I(x)$ and $I_\rho(x)$ are defined respectively by (46) and (71) and $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ such that $|\gamma| < T$.

Substituting the last expression for $Y(X)$ in (67), together with (70) and (72) gives

$$L(x) - \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} I(x) \ll X(\log X)^{3-B} + (\log X)^3 \mathfrak{K}, \quad (73)$$

where

$$\mathfrak{K} = \sum_{d \leq D} \frac{1}{d} \sum_{\chi \pmod{d}^*} \sum_{|\gamma| \leq T} |I_\rho(x)|. \quad (74)$$

Consider the integral $I_\rho(x)$ defined by (71) and let $\rho = \beta + i\gamma$. With a change of variables we can write the integral in the form

$$I_\rho(x) = \frac{1}{c} \int_{(\mu X)^c}^{X^c} u^{\frac{1}{c}\rho-1} e(xu) du = \int_{(\mu X)^c}^{X^c} u^{\frac{1}{c}\beta-1} e(h(u)) du,$$

where

$$h(u) = \frac{\gamma}{2\pi c} \log u + xu.$$

We have

$$h'(u) = \frac{\gamma}{2\pi c u} + x, \quad h''(u) = \frac{-\gamma}{2\pi c u^2}.$$

Suppose that $|\gamma| \geq 4\pi c|x|X^c$. Then we have $|h'(u)| \asymp \frac{|\gamma|}{X^c}$ and we estimate the integral using Lemma 5. We find that

$$I_\rho(x) \ll \frac{X^\beta}{|\gamma|} \quad \text{if} \quad 4\pi c|x|X^c \leq |\gamma| \leq T. \quad (75)$$

If $\pi c\mu^c|x|X^c < |\gamma| < 4\pi c|x|X^c$, then we have $|h''(u)| \asymp \frac{|\gamma|}{X^{2c}} \asymp \frac{|x|}{X^c}$ and using Lemma 5 we find that

$$I_\rho(x) \ll \frac{X^{\beta-c}}{\sqrt{|x|X^{-c}}} = \frac{X^\beta}{\sqrt{|x|X^c}}.$$

Finally, if $|\gamma| \leq \pi c\mu^c|x|X^c$, then we have $|h'(u)| \asymp |x|$ and applying again Lemma 5 we get

$$I_\rho(x) \ll \frac{X^\beta}{|x|X^c}.$$

However, from (64) it follows that $|x|X^c \geq \sqrt{|x|X^c}$, hence from the last two estimates we obtain

$$I_\rho(x) \ll \frac{X^\beta}{\sqrt{|x|X^c}} \quad \text{if} \quad |\gamma| < 4\pi c|x|X^c. \quad (76)$$

From (74) – (76) we find that

$$\mathfrak{K} \ll \mathfrak{K}' + \mathfrak{K}'', \quad (77)$$

where \mathfrak{K}' is the contribution coming from the terms for which the condition (75) for γ is satisfied and respectively, \mathfrak{K}'' comes from the terms for which γ satisfies the condition for γ given in (76).

Consider \mathfrak{K}' . We have

$$\mathfrak{K}' = \sum_{d \leq D} \frac{1}{d} \sum_{\chi \pmod{d}^*} \sum_{4\pi c|x|X^c \leq |\gamma| \leq T} \frac{X^\beta}{|\gamma|}.$$

We divide the sum over d into $O(\log X)$ sums in which d runs over an interval of the form $(Q, Q']$, where $Q' \leq \min(2Q, D)$. We proceed with the sum over γ in the same way. Hence, we obtain

$$\mathfrak{K}' \ll (\log X)^2 \max_{Q \in [1, D]} \max_{L \in [4\pi c|x|X^c, T]} \left((QL)^{-1} \mathfrak{L}(L, Q, X) \right), \quad (78)$$

where $\mathfrak{L}(L, Q, X)$ is the sum defined by (52).

Having in mind (7), (8), (64), (70) and since for the above sums we have $Q \leq D$ and $L \leq T$, we easily verify the condition $Q^2 L \leq X^{\frac{12}{25}}$ imposed in Lemma 9. Then using (54), (64) and (78), we find that

$$\mathfrak{K}' \ll X(\log X)^{22 - \frac{B}{2}}. \quad (79)$$

Consider now the quantity \mathfrak{K}'' for which we have

$$\mathfrak{K}'' \ll \frac{1}{\sqrt{|x|X^c}} \sum_{d \leq D} \frac{1}{d} \sum_{\chi \pmod{d}^*} \sum_{|\gamma| \leq 4\pi c|x|X^c} X^\beta.$$

Using (52), the estimate (54) from Lemma 9 and (64) we find that

$$\mathfrak{K}'' \ll \frac{\log X}{\sqrt{|x|X^c}} \max_{Q \in [1, D]} \left(Q^{-1} \mathfrak{L}(4\pi c|x|X^c, Q, X) \right) \ll X(\log X)^{22 - \frac{B}{2}}. \quad (80)$$

From (77), (79) and (80) we obtain

$$\mathfrak{K} \ll X(\log X)^{22 - \frac{B}{2}}.$$

We substitute this estimate for \mathfrak{K} in (73) and, as the constant B can be taken arbitrarily large, we see that the asymptotic formula (59) is correct also in the case (64). This proves the lemma. □

The next lemma is an analog of [16, Lemma 7].

Lemma 11. *Let τ and Ξ be defined by (8) and (9). Then for the sum $L(x)$ and for the integral $I(x)$, defined respectively by (44) and (46), we have*

$$\int_{|x| < \tau} |L(x)|^2 dx \ll X^{2-c} (\log X)^6, \quad (81)$$

$$\int_{|x| < \tau} |I(x)|^2 dx \ll X^{2-c} (\log X)^4, \quad (82)$$

$$\int_{|x| < \Xi} |L(x)|^2 dx \ll X \Xi (\log X)^6. \quad (83)$$

Proof: We shall only prove (81), the other inequalities can be proved likewise. Denote by J the integral on the left side of (81). We have

$$\begin{aligned} J &= \int_{|x| < \tau} L(x) L(-x) dx \\ &= \int_{|x| < \tau} \sum_{d_1 \leq D} \lambda(d_1) \sum_{\substack{\mu X < p_1 \leq X \\ d_1 | p_1 + 2}} (\log p_1) e(x p_1^c) \sum_{d_2 \leq D} \lambda(d_2) \sum_{\substack{\mu X < p_2 \leq X \\ d_2 | p_2 + 2}} (\log p_2) e(-x p_2^c) dx \\ &= \sum_{d_1, d_2 \leq D} \lambda(d_1) \lambda(d_2) \sum_{\substack{\mu X < p_1, p_2 \leq X \\ d_i | p_i + 2, i=1,2}} (\log p_1) (\log p_2) \int_{|x| < \tau} e(x(p_1^c - p_2^c)) dx \\ &\ll (\log X)^2 \sum_{d_1, d_2 \leq D} \sum_{\substack{\mu X < n_1, n_2 \leq X \\ d_i | n_i + 2, i=1,2}} \min\left(\tau, \frac{1}{|n_1^c - n_2^c|}\right). \end{aligned}$$

We change the order of summation and use the obvious inequality $uv \leq u^2 + v^2$ to get

$$\begin{aligned} J &\ll (\log X)^2 \sum_{\mu X < n_1, n_2 \leq X} \tau(n_1 + 2) \tau(n_2 + 2) \min\left(\tau, \frac{1}{|n_1^c - n_2^c|}\right) \\ &\ll (\log X)^2 \sum_{\mu X < n_1, n_2 \leq X} \tau^2(n_1 + 2) \min\left(\tau, \frac{1}{|n_1^c - n_2^c|}\right). \end{aligned}$$

Now we proceed as in the proof of [16, Lemma 7] and we also use the well-known inequality $\sum_{n \leq y} \tau^2(n) \ll y(\log y)^3$. In this way we prove (81) — we leave the details to the reader. \square

We shall find asymptotic formulae for the integrals $\Gamma_1^{(1)}$ and $\Gamma_4^{(1)}$ defined respectively by (35) and (39). We consier only $\Gamma_1^{(1)}$ because the study of $\Gamma_4^{(1)}$ is similar.

From (22), (46) and Lemma 10 we know that if $|x| < \tau$, then we have

$$L^\pm(x) = \mathfrak{N}^\pm I(x) + O(X(\log X)^{-A}),$$

where $A > 0$ is arbitrarily large. Let

$$\mathfrak{I}_0 = \int_{|x| < \tau} \Theta(x) e(-Nx) I(x)^3 dx. \quad (84)$$

We use the identity

$$\begin{aligned} L^-(x) (L^+(x))^2 &= \mathfrak{N}^- (\mathfrak{N}^+)^2 I(x)^3 + (L^-(x) - \mathfrak{N}^- I(x)) (\mathfrak{N}^+)^2 I(x)^2 \\ &\quad + L^-(x) (L^+(x) - \mathfrak{N}^+ I(x)) \mathfrak{N}^+ I(x) + L^-(x) L^+(x) (L^+(x) - \mathfrak{N}^+ I(x)) \end{aligned}$$

and the obvious estimate

$$\mathfrak{N}^\pm \ll \log X \quad (85)$$

to find that

$$\left| L^-(x) (L^+(x))^2 - \mathfrak{N}^- (\mathfrak{N}^+)^2 I(x)^3 \right| \ll X(\log X)^{3-A} \left(|I(x)|^2 + |L^-(x)|^2 + |L^+(x)|^2 \right).$$

From the above inequality, the estimate (16) with $a = \frac{7}{8}\Delta$, from (35), (84) and the estimates (81), (82) from Lemma 11 it follows that

$$\Gamma_1^{(1)} - \mathfrak{N}^- (\mathfrak{N}^+)^2 \mathfrak{I}_0 \ll \Delta X^{3-c} (\log X)^{9-A}. \quad (86)$$

Consider now the integral

$$\mathfrak{I} = \int_{-\infty}^{\infty} \Theta(x) e(-Nx) I(x)^3 dx. \quad (87)$$

Following the proof of [16, Lemma 6] we see that for a suitable $\mu \in (0, 1)$ depending on c we have

$$\mathfrak{I} \gg \Delta X^{3-c}. \quad (88)$$

We leave the easy verification to the reader.

Further, we have $\left| \frac{d}{dt}(xt^c) \right| \gg |x|X^{c-1}$ for $t \in [\mu X, X]$. Therefore, applying again Lemma 5 we find that $I(x) \ll |x|^{-1}X^{1-c}$. From this estimate, (8), (16) with $a = \frac{7}{8}\Delta$, (84) and (87) we find that

$$|\mathfrak{I} - \mathfrak{I}_0| \leq \int_{|x| > \tau} |\Theta(x)| |I(x)|^3 dx \ll \Delta X^{3-3c} \int_{x > \tau} \frac{dx}{x^3} \ll \Delta X^{3-3c} \tau^{-2} \ll \Delta X^{3-c-2\xi}. \quad (89)$$

We apply (86) (with $A = 12$) as well as (85) and (89) to find

$$\Gamma_1^{(1)} = \mathfrak{N}^- (\mathfrak{N}^+)^2 \mathfrak{I} + O(\Delta X^{3-c} (\log X)^{-4}). \quad (90)$$

We proceed with $\Gamma_4^{(1)}$ is the same way and prove that

$$\Gamma_4^{(1)} = (\mathfrak{N}^+)^3 \mathfrak{I} + O(\Delta X^{3-c} (\log X)^{-4}). \quad (91)$$

4 The estimation of $\Gamma_1^{(2)}$ and $\Gamma_4^{(2)}$ and the end of the proof

From (43) we see that in order to find a non-trivial lower bound for Γ we have to prove that the integrals $\Gamma_1^{(2)}$ and $\Gamma_4^{(2)}$ are small enough. To establish this we need estimates for the sums $L^\pm(x)$ provided that $\tau \leq |x| \leq \Xi$.

We apply the next lemma, which is a special case of Vaughan's identity.

Lemma 12. *Let $f(n)$ be a complex valued function defined for integers $n \in (\mu X, X]$. Then we have*

$$\sum_{\mu X < n \leq X} \Lambda(n) f(n) = S_1 - S_2 - S_3,$$

where

$$S_1 = \sum_{k \leq X^{\frac{1}{3}}} \mu(k) \sum_{\frac{\mu X}{k} < l \leq \frac{X}{k}} (\log l) f(kl), \quad (92)$$

$$S_2 = \sum_{k \leq X^{\frac{2}{3}}} c(k) \sum_{\frac{\mu X}{k} < l \leq \frac{X}{k}} f(kl), \quad (93)$$

$$S_3 = \sum_{X^{\frac{1}{3}} < k \leq X^{\frac{2}{3}}} a(k) \sum_{\frac{\mu X}{k} < l \leq \frac{X}{k}} \Lambda(l) f(kl) \quad (94)$$

and where $a(k)$, $c(k)$ are real numbers satisfying

$$|a(k)| \leq \tau(k), \quad |c(k)| \leq \log k. \quad (95)$$

Proof. Can be found in [20].

□

Follows Van der Corput's inequality.

Lemma 13. *Let α, β be real numbers with $\beta - \alpha \geq 1$ and let H be a positive integer. Suppose that for any integer $l \in (\alpha, \beta]$ there is a complex number $\Upsilon(l)$. Then we have*

$$\left| \sum_{\alpha < l \leq \beta} \Upsilon(l) \right|^2 \leq \frac{\beta - \alpha + H}{H} \sum_{|h| < H} \left(1 - \frac{|h|}{H} \right) \sum_{\alpha < l, l+h \leq \beta} \Upsilon(l+h) \overline{\Upsilon(l)}. \quad (96)$$

Proof. Can be found in [8, Lemma 8.17].

□

The next lemma presents Van der Corput's estimate for exponential sums.

Lemma 14. Suppose that α, β are real numbers with $\beta - \alpha \geq 1$ and let $f(y)$ be two times continuously differentiable function in the interval $[\alpha, \beta]$. Assume also that for some $\lambda > 0$ we have $|f''(y)| \asymp \lambda$ uniformly for $y \in [\alpha, \beta]$. Then we have

$$\left| \sum_{\alpha < n \leq \beta} e(f(n)) \right| \ll (\beta - \alpha) \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}.$$

Proof. See [9, Chapter 1, Theorem 5]. □

From this point onwards we assume that

$$\xi = \frac{459c - 435}{125}, \quad \delta = \frac{180 - 168c}{125}. \quad (97)$$

(It is easy to verify that (97) implies the first inequality from (7)). We prove the following

Lemma 15. Suppose that D, τ and Ξ are defined by (8) and (9) and ξ, δ are specified by (97). Suppose also that the real numbers $\lambda(d)$ satisfy (45) and $L(x)$ is defined by (44). Then there exists $\varkappa(c) > 0$ such that

$$\sup_{\tau \leq |x| \leq \Xi} |L(x)| \ll X^{2-c-\varkappa(c)}. \quad (98)$$

Proof. Instead of $L(x)$ we consider the sum $L_1(x)$ given by (61) and we take into account (60). We write $L_1(x)$ in the form

$$L_1(x) = \sum_{\mu X < n \leq X} \Lambda(n) f(n), \quad f(n) = \sum_{\substack{d \leq D \\ d|n+2}} \lambda(d) e(xn^c) \quad (99)$$

and we apply Lemma 12 to find that

$$L_1(x) = S_1 - S_2 - S_3, \quad (100)$$

where S_1, S_2 and S_3 are the sums defined respectively by (92) – (94) with the function $f(n)$ given by (99).

From (93) we find that

$$S_2 = S'_2 + S''_2, \quad (101)$$

where

$$S'_2 = \sum_{k \leq X^{\frac{1}{3}}} c(k) \sum_{\frac{\mu X}{k} < l \leq \frac{X}{k}} f(kl), \quad (102)$$

$$S''_2 = \sum_{X^{\frac{1}{3}} < k \leq X^{\frac{2}{3}}} c(k) \sum_{\frac{\mu X}{k} < l \leq \frac{X}{k}} f(kl). \quad (103)$$

Therefore we have

$$L_1(x) \ll |S_1| + |S'_2| + |S''_2| + |S_3|. \quad (104)$$

Consider the sum S'_2 . We use (99) and (102) and change the order of summation to write it in the form

$$S'_2 = \sum_{d \leq D} \lambda(d) \sum_{k \leq X^{\frac{1}{3}}} c(k) \sum_{\substack{\frac{\mu X}{k} < l \leq \frac{X}{k} \\ d|kl+2}} e(x(kl)^c).$$

Since $\lambda(d) = 0$ for $2 \mid d$, then from $d \mid kl + 2$ it follows that $(k, d) = 1$. Hence there exists an integer l_0 such that $d \mid kl + 2$ is equivalent to $l \equiv l_0 \pmod{d}$, which means that $l = l_0 + md$ for some integer m . Therefore we get

$$S'_2 = \sum_{d \leq D} \lambda(d) \sum_{\substack{k \leq X^{\frac{1}{3}} \\ (k, d) = 1}} c(k) \sum_{\frac{\mu X}{kd} - \frac{l_0}{d} < m \leq \frac{X}{kd} - \frac{l_0}{d}} e(h(m)), \quad h(m) = xk^c(l_0 + md)^c. \quad (105)$$

We have $h''(m) = c(c-1)xk^c d^2(l_0 + md)^{c-2}$, hence $|h''(m)| \asymp |x|k^2 d^2 X^{c-2}$ and using Lemma 14 we find that the sum over m in (105) is

$$\ll \frac{X}{kd} (|x|k^2 d^2 X^{c-2})^{\frac{1}{2}} + (|x|k^2 d^2 X^{c-2})^{-\frac{1}{2}} \ll |x|^{\frac{1}{2}} X^{\frac{c}{2}} + |x|^{-\frac{1}{2}} (kd)^{-1} X^{1-\frac{c}{2}}.$$

Then using (45), (95) and (105) we find that

$$S'_2 \ll X^\varepsilon \left(D|x|^{\frac{1}{2}} X^{\frac{1}{3}+\frac{c}{2}} + |x|^{-\frac{1}{2}} X^{1-\frac{c}{2}} \right). \quad (106)$$

To estimate S_1 we apply Abel's summation formula to get rid of the factor $(\log l)$ in the sum from (92) and then proceed as above. In this way we find that

$$S_1 \ll X^\varepsilon \left(D|x|^{\frac{1}{2}} X^{\frac{1}{3}+\frac{c}{2}} + |x|^{-\frac{1}{2}} X^{1-\frac{c}{2}} \right). \quad (107)$$

We leave the simple calculations to the reader.

Consider now the sum S_3 . We divide it into $O(\log X)$ sums of type

$$W(K) = \sum_{K < k \leq K_1} a(k) \sum_{\substack{\frac{\mu X}{k} < l \leq \frac{X}{k}}} \Lambda(l) \sum_{\substack{d \leq D \\ d|kl+2}} \lambda(d) e(x(kl)^c), \quad (108)$$

where

$$K_1 \leq 2K, \quad X^{\frac{1}{3}} \leq K < K_1 \leq X^{\frac{2}{3}}. \quad (109)$$

Consider the case

$$X^{\frac{1}{2}} \leq K. \quad (110)$$

It is clear that

$$W(K) \ll X^\varepsilon \sum_{K < k \leq K_1} \left| \sum_{\substack{\frac{\mu X}{k} < l \leq \frac{X}{k}}} \Upsilon(l) \right|, \quad (111)$$

where

$$\Upsilon(l) = \Lambda(l) \sum_{\substack{d \leq D \\ d|kl+2}} \lambda(d) e(x(kl)^c). \quad (112)$$

Applying Cauchy's inequality, we find that

$$|W(K)|^2 \ll X^\varepsilon K \sum_{K < k \leq K_1} \left| \sum_{\substack{\frac{\mu X}{k} < l \leq \frac{X}{k}}} \Upsilon(l) \right|^2. \quad (113)$$

Assume that H is an integer satisfying

$$1 \leq H \ll \frac{X}{K} \quad (114)$$

and let $\Upsilon(l)$ be defined by (112). For the inner sum in (113) we apply the inequality (96) from Lemma 13 and we find that

$$|W(K)|^2 \ll \frac{X^{1+\varepsilon}}{H} \sum_{K < k \leq K_1} \sum_{|h| < H} \left(1 - \frac{|h|}{H}\right) \sum_{\substack{\frac{\mu X}{k} < l, l+h \leq \frac{X}{k}}} \Lambda(l+h) \sum_{\substack{d_2 \leq D \\ d_2|k(l+h)+2}} \lambda(d_2) e(x(k(l+h))^c) \Lambda(l) \sum_{\substack{d_1 \leq D \\ d_1|kl+2}} \lambda(d_1) e(-x(kl)^c).$$

We change the order of summation and find

$$|W(K)|^2 \ll \frac{X^{1+\varepsilon}}{H} \sum_{d_1, d_2 \leq D} \lambda(d_1) \lambda(d_2) \sum_{|h| < H} \left(1 - \frac{|h|}{H}\right) \sum_{\substack{\frac{\mu X}{K_1} < l, l+h \leq \frac{X}{K}}} \Lambda(l) \Lambda(l+h) \mathfrak{F}, \quad (115)$$

where

$$\mathfrak{F} = \sum_{\substack{\widetilde{K} < k \leq \widetilde{K}_1 \\ d_1|kl+2 \\ d_2|k(l+h)+2}} e(xk^c((l+h)^c - l^c))$$

and

$$\widetilde{K} = \max\left(K, \frac{\mu X}{l}, \frac{\mu X}{l+h}\right), \quad \widetilde{K}_1 = \min\left(K_1, \frac{X}{l}, \frac{X}{l+h}\right).$$

Since $\lambda(d) = 0$ for $2 \mid d$, we may assume that $2 \nmid d_1 d_2$. Hence $(d_1, l) = (d_2, l+h) = 1$ because otherwise the sum \mathfrak{F} would be empty. Therefore, there exists an integer k_0 depending on l, h, d_1, d_2 such that the pair of conditions $d_1 \mid kl+2$ and $d_2 \mid k(l+h)+2$ is equivalent to the congruence $k \equiv k_0 \pmod{[d_1, d_2]}$. Hence we may write the sum \mathfrak{F} as

$$\mathfrak{F} = \sum_{\substack{\frac{\widetilde{K}-k_0}{[d_1, d_2]} < m \leq \frac{\widetilde{K}_1-k_0}{[d_1, d_2]}} e(F(m)),$$

where

$$F(m) = x \left((l+h)^c - l^c \right) (k_0 + m[d_1, d_2])^c.$$

Obviously, we have

$$\mathfrak{F} \ll \frac{K}{[d_1, d_2]} \quad \text{if} \quad h = 0. \quad (116)$$

Consider now the case $h \neq 0$. We have

$$F''(m) = c(c-1)x \left((l+h)^c - l^c \right) (k_0 + m[d_1, d_2])^{c-2} [d_1, d_2]^2,$$

hence

$$|F''(m)| \asymp |x| |(l+h)^c - l^c| K^{c-2} [d_1, d_2]^2.$$

We apply Lemma 14 and find that

$$\begin{aligned} \mathfrak{F} &\ll \frac{K}{[d_1, d_2]} \left(|x| |(l+h)^c - l^c| K^{c-2} [d_1, d_2]^2 \right)^{\frac{1}{2}} + \left(|x| |(l+h)^c - l^c| K^{c-2} [d_1, d_2]^2 \right)^{-\frac{1}{2}} \\ &\ll |x|^{\frac{1}{2}} K^{\frac{c}{2}} |(l+h)^c - l^c|^{\frac{1}{2}} + |x|^{-\frac{1}{2}} K^{1-\frac{c}{2}} |(l+h)^c - l^c|^{-\frac{1}{2}} [d_1, d_2]^{-1}. \end{aligned}$$

We use the above estimate, (115), (116) as well as the estimate

$$\sum_{d_1, d_2 \leq D} \frac{1}{[d_1, d_2]} \ll (\log X)^3$$

(we leave the easy verification to the reader) to obtain

$$|W(K)|^2 \ll X^{2+\varepsilon} H^{-1} + X^{1+\varepsilon} D^2 |x|^{\frac{1}{2}} K^{\frac{c}{2}} H^{-1} \Sigma_1 + X^{1+\varepsilon} |x|^{-\frac{1}{2}} K^{1-\frac{c}{2}} H^{-1} \Sigma_1,$$

where

$$\begin{aligned} \Sigma_1 &= \sum_{0 < |h| \leq H} \sum_{\frac{\mu X}{K_1} < l+h \leq \frac{X}{K}} |(l+h)^c - l^c|^{\frac{1}{2}}, \\ \Sigma_2 &= \sum_{0 < |h| \leq H} \sum_{\frac{\mu X}{K_1} < l+h \leq \frac{X}{K}} |(l+h)^c - l^c|^{-\frac{1}{2}}. \end{aligned}$$

By a straightforward calculation, which we leave to the reader, one obtains

$$\Sigma_1 \ll H^{\frac{3}{2}} X^{\frac{c+1}{2}} K^{-\frac{c+1}{2}}, \quad \Sigma_2 \ll H^{\frac{1}{2}} X^{\frac{3-c}{2}} K^{-\frac{3-c}{2}}.$$

Therefore, we find

$$|W(K)|^2 \ll X^\varepsilon \left(X^2 H^{-1} + X^{\frac{3+c}{2}} D^2 |x|^{\frac{1}{2}} K^{-\frac{1}{2}} H^{\frac{1}{2}} + X^{\frac{5-c}{2}} |x|^{-\frac{1}{2}} K^{-\frac{1}{2}} H^{-\frac{1}{2}} \right). \quad (117)$$

We choose

$$H = \left[\min(H_0, X K^{-1}) \right], \quad \text{where} \quad H_0 = X^{\frac{1-c}{3}} K^{\frac{1}{3}} D^{-\frac{4}{3}} |x|^{-\frac{1}{3}}. \quad (118)$$

(It is easy to verify that (114) holds). We note that

$$H^{-1} \asymp H_0^{-1} + KX^{-1}. \quad (119)$$

Using (109), (110) and (117) – (119) we obtain

$$W(K) \ll X^\varepsilon \left(X^{\frac{3}{4}+\frac{\varepsilon}{6}} D^{\frac{2}{3}} |x|^{\frac{1}{6}} + X^{\frac{5}{6}} + X^{1-\frac{\varepsilon}{6}} D^{\frac{1}{3}} |x|^{-\frac{1}{6}} + X^{1-\frac{\varepsilon}{4}} |x|^{-\frac{1}{4}} \right). \quad (120)$$

Consider now the sum $W(K)$ defined by (108) in the case

$$K < X^{\frac{1}{2}}. \quad (121)$$

We write it in the form

$$W(K) = \sum_{\frac{\mu X}{K_1} < l \leq \frac{X}{K}} \Lambda(l) \sum_{\max(K, \frac{\mu X}{l}) < k \leq \min(K_1, \frac{X}{l})} a(k) \sum_{\substack{d \leq D \\ d|kl+2}} \lambda(d) e(x(kl)^c).$$

Now we have $\frac{X}{K} \gg X^{\frac{1}{2}}$ and we may proceed as above but with rôles of k and l reversed. Finally, we establish again the estimate (120).

Since the sum S_3 consists of $O(\log X)$ sums of type $W(K)$, it can be estimated by the expression from the right side of (120), too. We study the sum S_2'' in the same manner and we obtain

$$S_2'', S_3 \ll X^\varepsilon \left(X^{\frac{3}{4}+\frac{\varepsilon}{6}} D^{\frac{2}{3}} |x|^{\frac{1}{6}} + X^{\frac{5}{6}} + X^{1-\frac{\varepsilon}{6}} D^{\frac{1}{3}} |x|^{-\frac{1}{6}} + X^{1-\frac{\varepsilon}{4}} |x|^{-\frac{1}{4}} \right). \quad (122)$$

From (60), (104), (106), (107) and (122) we find that

$$L(x) \ll X^\varepsilon \left(X^{\frac{1}{3}+\frac{\varepsilon}{2}} D |x|^{\frac{1}{2}} + X^{1-\frac{\varepsilon}{2}} |x|^{-\frac{1}{2}} + X^{\frac{3}{4}+\frac{\varepsilon}{6}} D^{\frac{2}{3}} |x|^{\frac{1}{6}} + X^{\frac{5}{6}} + X^{1-\frac{\varepsilon}{6}} D^{\frac{1}{3}} |x|^{-\frac{1}{6}} + X^{1-\frac{\varepsilon}{4}} |x|^{-\frac{1}{4}} \right).$$

Now we use (8) to find that if $\tau \leq |x| \leq \Xi$, then we have $X^{\xi-c} \leq |x| \ll X^\varepsilon$. Hence

$$L(x) \ll X^\varepsilon \left(X^{\frac{1}{3}+\frac{\varepsilon}{2}+\delta} + X^{\frac{3}{4}+\frac{\varepsilon}{6}+\frac{2\delta}{3}} + X^{1-\frac{\varepsilon}{6}+\frac{\delta}{3}} + X^{1-\frac{\varepsilon}{4}} \right).$$

It remains to use (2) and (97) and after a simple calculation we obtain (98). □

We are now in position to estimate the quantities $\Gamma_1^{(2)}$ and $\Gamma_4^{(2)}$ defined respectively by (36) and (40). We apply Lemma 15 and use (16) with $a = \frac{7}{8}\Delta$ to find that

$$\Gamma_1^{(2)}, \Gamma_4^{(2)} \ll X^{2-c-\kappa(c)} \Delta \int_{|x| \leq \Xi} |L^+(x)|^2 dx.$$

From the above formula, the definitions of Δ and Ξ (see (3) and (9)) and formula (83) of Lemma 11 we conclude that

$$\Gamma_1^{(2)}, \Gamma_4^{(2)} \ll \Delta X^{3-c} (\log X)^{-4}.$$

From the last formula and (43), (88), (90), (91) we conclude that

$$\Gamma \geq |3\mathfrak{N}^- - 2\mathfrak{N}^+| (\mathfrak{N}^+)^2 \mathfrak{J} + O(\Delta X^{3-c}(\log X)^{-4}). \quad (123)$$

Now we shall find a lower bound for the difference $3\mathfrak{N}^- - 2\mathfrak{N}^+$. It is clear that for the quantity \mathfrak{P} defined by (22) we have

$$\mathfrak{P} \asymp (\log X)^{-1}. \quad (124)$$

From (23) and (24) we see that

$$3\mathfrak{N}^- - 2\mathfrak{N}^+ \geq \mathfrak{P} (3f(s_0) - 2F(s_0)) + O((\log X)^{-\frac{1}{3}}), \quad (125)$$

where s_0 is defined by (22) and $F(s)$, $f(s)$ are the functions specified by (25). We take $s_0 = 2,95$ which means that

$$\eta = \frac{\delta}{2,95} = \frac{180 - 168c}{368,75}.$$

Hence, using (25) we see that $3f(s_0) - 2F(s_0) > 0$.

It remains to take into account the lower bound for \mathfrak{N}^+ in (23) as well as (88), (124), (125) to obtain

$$\Gamma \gg \Delta X^{3-c}(\log X)^{-3}.$$

Therefore $\Gamma > 0$ and the equation (1) with Δ specified by (3) has a solution in primes p_1, p_2, p_3 such that each of the numbers $p_1 + 2, p_2 + 2, p_3 + 2$ has at most $\left\lfloor \frac{369}{180-168c} \right\rfloor$ prime factors. This completes the proof of Theorem 1. □

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